

Successful Manipulation in Stable Marriage Model with Complete Preference Lists

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Abstract. This paper deals with a strategic issue in the stable marriage model with complete preference lists (i.e., a preference list of an agent is a permutation of all the members of the opposite sex). Given complete preference lists of n men over n women, and a marriage μ , we consider the problem for finding preference lists of n women over n men such that the men-proposing deferred acceptance algorithm (Gale-Shapley algorithm) adopted to the lists produces μ . We show a simple necessary and sufficient condition for the existence of a set of preference lists of women over men. Our condition directly gives an $O(n^2)$ time algorithm for finding a set of preference lists, if it exists.

1 Introduction

In 1962, Gale and Shapley [1] proposed a simple model, called *stable marriage model*, of two-sided matching market. Given two sets of agents, men and women, both of size n , Gale and Shapley discussed a model in which each agent had preferences over agents of the opposite sex. A marriage is a one-to-one mapping between the two sexes such that a man m is mapped to a woman w , if and only if, w is mapped to m . A marriage is called stable, if no man and woman who are not mapped to each other would both prefer to be. Gale and Shapley proposed an algorithm, called deferred acceptance algorithm (Gale-Shapley algorithm), which *always* finds a stable marriage.

The deferred acceptance algorithm is employed in a number of labor market clearinghouses and college admission systems. A notable variation of the algorithm, called men-proposing deferred acceptance algorithm, works by having men make proposals to women and produces men-optimal marriage, which every man likes at least as well as any other stable marriage. Since most applications of the deferred acceptance algorithm involve the participation of independent agents, it is natural to ask whether agents can benefit by being dishonest about their preference lists. It is well-known that stating true preferences is a dominant strategy for the men in men-proposing deferred acceptance algorithm. In settings that allow incomplete preference lists, women on the other hand have

incentives to submit false preferences. By contrary, little is known in the case of stable marriage model with complete preference lists.

This paper deals with a strategic issue in the stable marriage model with complete preference lists (i.e., a preference list of an agent is a permutation of all the members of the opposite sex). Given complete preference lists of n men over n women, and a marriage μ , we consider the problem for finding complete preference lists of n women over n men such that the men-proposing deferred acceptance algorithm adopted to the lists produces μ . We show a simple necessary and sufficient condition for the existence of a set of complete preference lists of women over men. Our condition directly gives an $O(n^2)$ time algorithm for finding a set of preference lists, if it exists.

In the next section, we establish some terminology and definitions and give some background. Section 3 gives our main results.

2 Notations and Definitions

We denote two sets of agents by M and W , called men and women, both of size n . Each agent in $M \cup W$ has a *preference list* which is a totally ordered list of all the members of the opposite sex. Here we note that this paper considers the case with ‘complete’ preference lists, i.e., a preference list of an agent is a permutation of members of the opposite sex. A *marriage* is a mapping $\mu : (M \cup W) \rightarrow (M \cup W)$ satisfying that (1) $\forall m \in M, \mu(m) \in W$, (2) $\forall w \in W, \mu(w) \in M$, and (3) $w = \mu(m)$ if and only if $m = \mu(w)$. For any agent $i \in M \cup W$, $\mu(i)$ is called the *mate* of i in marriage μ . A pair $(m, w) \in M \times W$ is called a *blocking pair* for a marriage μ , if m prefers w to $\mu(m)$ and w prefers m to $\mu(w)$. A marriage with no blocking pair is called a *stable marriage*.

Gale and Shapley [1] showed that a stable marriage always exists, and a simple algorithm called the *deferred acceptance algorithm* can find a stable marriage. Here we briefly describe a variant of the their algorithm in which men propose to women (these roles can naturally be reversed). In the following algorithm, we introduced an iteration number which will be used in a later section.

Men-Proposing Deferred Acceptance Algorithm

Step 0: Set the iteration number $r := 1$ and unmarried men $U := M$.

Initially, every woman has no current mate.

Step 1: If $U = \emptyset$, then output the current mate of every woman and stop.

Step 2: Choose a man $m \in U$. Let $w \in W$ be m 's most preferred woman who hasn't yet rejected m .

Step 3: (Create a proposal from man m to woman w .)

If woman w has no mate, then update $U := U \setminus \{m\}$ and set m be the current mate of w .

Else if w prefers m to her current mate m' , then w rejects m' , update $U := U \setminus \{m'\} \cup \{m\}$ and set m be the current mate of w .

Else, w rejects m .

Step 4: Update $r := r + 1$ and go to Step 1.

It is known that the order of proposals (choice of $m \in U$ in Step 2) does not affect the output of the algorithm [1]. Conway showed that the set of stable marriages can be partially ordered as a lattice with the pair of extremal elements, called men-optimal and women-optimal marriages (see [9] for example). In fact, men-proposing deferred acceptance algorithm produces men-optimal marriage.

The issues of strategic manipulation in the stable marriage are discussed in many papers (see books [4, 6] and the references therein, for example). Roth [5] showed that when the men-proposing algorithm is used, none of the men benefits by submitting a false preference list, regardless of how the other agents report their preference. Dubins and Freedman [10] proved that no coalition of men could collectively manipulate in such a way as to strictly improve all of their mates in comparison to men-optimal marriage. In settings that allow incomplete preference lists, women on the other hand have incentives to cheat in men-proposing algorithm. Gale and Sotomayor [2] showed that a woman has an incentive to falsify their preferences as long as she has at least two distinct stable mates. In fact, the women can force the women-optimal marriage μ by rejecting all the men except mates in μ (see [3]).

A feature of this paper is that the agents are required to submit complete preference list. Comparing to the above results, little is known in the case of stable marriage model with complete preference lists. Gusfield and Irving ([4], page 65) point to the absence of any general results in this setting. Tadenuma and Toda [8] considered an implementation question. Teo, Sethuraman, and Tan [7] deals with the situation that there exists a specified woman w who is the only deceitful agent, and that she knows the reported preferences of all the other agents. They proposed a polynomial time algorithm for constructing woman w 's optimal cheating strategy. They also discussed the Singapore school-admissions problem, where stable marriage model with complete preference lists is a suitable representation of the problem.

3 Main Results

In this paper, we consider the following problem.

Problem P(\mathcal{L}^m, μ):

Input: Set of preference lists \mathcal{L}^m of men M over women W and a marriage μ .

Question: If there exists a set of preference lists \mathcal{L}^w of women W over men M such that the men-proposing deferred acceptance algorithm adopted to the lists in \mathcal{L}^m and \mathcal{L}^w produces the marriage μ , then output \mathcal{L}^w . If not, say ‘none exists.’

We give a simple necessary and sufficient condition for the existence of women's lists. Let $G(\mathcal{L}^m, \mu)$ be a directed bipartite graph with a pair of vertex sets M, W and a set of directed edges A defined by

$$A = \{(w, \mu(w)) \in W \times M \mid w \in W\} \\ \cup \left\{ (m, w) \in M \times W \mid \begin{array}{l} w = \mu(m) \text{ or} \\ m \text{ prefers } w \text{ to } \mu(m) \end{array} \right\}.$$

Here we note that for any pair of mates $\{i, \mu(i)\}$ in μ , there are parallel directed edges with opposite directions between corresponding vertices in $G(\mathcal{L}^m, \mu)$.

A directed graph is said to be *strongly connected* if, for all pair of vertices i and j , there exists a directed path from i to j . A *strong component* of a directed graph is a strongly connected subgraph which is maximal. In this paper, we denote a strong component by a set of vertices in the corresponding strongly connected subgraph. If a strong component V' has no incoming edge, i.e., every edge connecting vertices $i \in V'$ and $j \notin V'$ is incident from i to j , we say that V' is a *minimal strong component*.

Theorem 1. *Let \mathcal{L}^m be a given set of preference lists of men M over women W , and μ a given marriage. There exists a set of preference lists \mathcal{L}^w of women over men, such that the men-proposing deferred acceptance algorithm adopted to the lists in \mathcal{L}^m and \mathcal{L}^w produces the marriage μ , if and only if, every minimal strong component of $G(\mathcal{L}^m, \mu)$ consists of exactly one pair of vertices.*

Before proving the above theorem, we describe some properties of strong components in $G(\mathcal{L}^m, \mu)$. Since every pair of vertices $\{i, \mu(i)\}$ induces a strongly connected subgraph of $G(\mathcal{L}^m, \mu)$, vertices i and $\mu(i)$ are contained in a common strong component of $G(\mathcal{L}^m, \mu)$ for any agent $i \in M \cup W$. Thus, for any strong component V' in $G(\mathcal{L}^m, \mu)$, there exists a subset of men $M' \subseteq M$ satisfying that $V' = \cup_{m \in M'} \{m, \mu(m)\}$. Clearly, the equality $|V' \cap M| = |V' \cap W|$ holds. Every vertex $w \in W$ has a unique outgoing edge $(w, \mu(w))$ and every vertex $m \in M$ has a unique incoming edge $(\mu(m), m)$. These properties yield that a pair of vertices $m \in M$ and $\mu(m) \in W$ forms a minimal strong component, if and only if, vertex $\mu(m)$ has a unique incoming edge $(m, \mu(m))$. Lastly, we note that the strong component decomposition of $G(\mathcal{L}^m, \mu)$ is essentially equivalent to the Dulmage-Mendelsohn decomposition of corresponding underlying undirected graph [11].

Proof. First, consider the case that there exists a minimal strong component V' of $G(\mathcal{L}^m, \mu)$ including more than two vertices. From the assumption, there exists a subset of men $M' \subseteq M$ satisfying that $|M'| \geq 2$ and $V' = \cup_{m \in M'} \{m, \mu(m)\}$.

Let \mathcal{L}^w be an arbitrary set of preference lists of W over M . We apply the men-proposing deferred acceptance algorithm to lists in \mathcal{L}^m and \mathcal{L}^w and assume on the contrary that the marriage μ is obtained. For any $m \in M$, $r(m)$ denotes the iteration number when man $m \in M$ proposed to $\mu(m)$. Let m^* be a man in M' who proposed to his mate in μ lastly, i.e., m^* satisfies $r(m^*) = \max_{m \in M'} r(m)$. We denote $r(m^*)$ by r^* for simplicity.

At the beginning of r^* th iteration, man m^* is unmarried and every man $m \in M' \setminus \{m^*\}$ is the current mate of women $\mu(m)$. Now we show that women $\mu(m^*)$ also has a current mate at the beginning of r^* th iteration. We denote $\mu(m^*)$ by w^* .

Since $\{m^*, w^*\}$ is not a strong component, vertex w^* has an incoming edge (m', w^*) different from the edge (m^*, w^*) . The minimality of V' yields that the vertex m' is contained in $M' \setminus \{m^*\}$. From the definition of the graph, m' prefers w^* to its mate $\mu(m')$ and thus w^* has rejected m' in an iteration earlier than r^* . In the deferred acceptance algorithm, if a women rejects a man, she holds a

current mate in the rest of iterations. Thus women w^* has a current mate at the beginning of r^* -th iteration.

From the above, all the women in V' have current mates, denoted by M'' , at the beginning of r^* -th iteration. Since m^* is unmarried at the beginning of r^* -th iteration, $M'' \subseteq M' \setminus \{m^*\}$. Thus, we have that

$$\begin{aligned} |V' \cap W| &= |M''| \leq |M' \setminus \{m^*\}| < |M'| \\ &= |V' \cap M| = |V' \cap W|. \end{aligned}$$

Contradiction.

Next, we show the inverse implication. Let us consider the case that every minimal strong component of $G(\mathcal{L}^m, \mu)$ consists of exactly one pair of vertices. First, we modify the directed graph as follows. We introduce an artificial vertex s and add directed edge (s, w) for each (woman) vertex $w \in W$ satisfying that w is contained in a minimal strong component. For any vertex $i \in M \cup W$, there exists a directed path from s to i . Thus there exists a directed outgoing spanning tree, denoted by T , with root vertex s in the modified graph. (We fix a directed outgoing spanning tree T in the rest of this proof.) For each vertex $i \in M \cup W$, we denote the parent vertex of i in T by $\text{prt}(i)$. Here we note that for any man (vertex) $m \in M$, his parent vertex $\text{prt}(m)$ is equivalent to his mate $\mu(m)$. The parent vertex of a woman (vertex) is either the artificial vertex s or a man (vertex).

Now we construct preference lists \mathcal{L}^w of women as follows. For any woman w contained in a minimal strong component, we employ a preference list (a total order of men M) such that the most preferred man is $\mu(w)$. If a woman w is not contained in any minimal strong component, we adopt a preference list (a total order of men M) of w such that woman w 's first choice is $\mu(w)$ and her second choice is $\text{prt}(w)$ in T .

We apply the men-proposing deferred acceptance algorithm to lists in \mathcal{L}^m and \mathcal{L}^w . Since each woman w most prefers man $\mu(w)$ in the list \mathcal{L}^w , w never rejects man $\mu(w)$ in the algorithm. Thus, if man m proposed to woman w in the algorithm, then $w = \mu(m)$ or m prefers w to $\mu(m)$, and consequently, the graph $G(\mathcal{L}^m, \mu)$ includes the directed edge (m, w) .

Let μ' be a marriage obtained by the men-proposing deferred acceptance algorithm adopted to lists in \mathcal{L}^m and \mathcal{L}^w . In the rest of this proof, we show that $\mu = \mu'$ by induction on heights of vertices defined below. For any vertex $i \in M \cup W$, $h(i)$ denotes the *height* of i in T , i.e., $h(i)$ is equal to the length of a unique path from s to i in T . We define that $h(s) = 0$.

(1) Let $i \in M \cup W$ be an agent with $h(i) = 1$. Clearly from the definition of the modified graph, i is a woman contained in a minimal strong component and has exactly two incoming edges $\{(s, i), (\mu(i), i)\}$ in the modified graph. Since man $\mu'(i)$ proposed to i in the algorithm, there is a directed edge $(\mu'(i), i)$ in $G(\mathcal{L}^m, \mu)$. From the above, we have that $\mu(i) = \mu'(i)$.

(2) Assume that for any vertex $j \in M \cup W$, $h(j) = h'$ yields $\mu(j) = \mu'(j)$. Let $i \in M \cup W$ be a vertex whose height is $h' + 1$, i.e., $h(i) = h' + 1$.

(2-1) If $h' + 1$ is an even number, i corresponds to a man, denoted by $m \in M$. Since the vertex m has a unique incoming edge $(\mu(m), m)$, $\mu(m)$ is the parent vertex of m , whose height is h' . The assumption of induction yields that $\mu'(\mu(m)) = \mu(\mu(m)) = m$. From the definition of marriage, $\mu'(\mu(m)) = m$ implies that $\mu(m) = \mu'(m)$.

(2-2) If $h' + 1$ is an odd number, i corresponds to a woman $w \in W$. We denote w 's parent vertex $\text{prt}(w)$ by $m' \in M$, for simplicity. From the assumption of induction, $\mu'(m') = \mu(m')$ and thus man m' proposed to woman $\mu(m')$ in the algorithm. Since the graph includes the directed edge (m', w) , man m' prefers w to $\mu(m')$. Consequently, man m' proposed to woman w and w rejected m' in the algorithm. In the preference list \mathcal{L}^W , man m' is w 's second choice. Since w rejected m' , w 's mate obtained in the algorithm is her first choice $\mu(w)$. Thus, we obtained a desired result that w 's mate obtained in the algorithm, denoted by $\mu'(w)$, is equivalent to $\mu(w)$. \square

The above theorem directly implies that we can solve the problem $P(\mathcal{L}^M, \mu)$ by constructing the strong component decomposition of the directed graph $G(\mathcal{L}^M, \mu)$, which requires $O(n^2)$ time [12].

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