Sealed Bid Multi-object Auctions with Necessary Bundles and its Application to Spectrum Auctions

Tomomi Matsui[1] and Takahiro Watanabe[2]

[1] Department of Mathematical Informatics,
Graduate School of Information Science and Technology,
University of Tokyo, Bunkyo-ku, Tokyo 113-0033, Japan,
tomomi@misojiro.t.u-tokyo.ac.jp,
WWW home page: http://www.misojiro.t.u-tokyo.ac.jp/~tomomi/
[2] Department of Policy Studies, Iwate Prefectural University,
twatanab@iwate-pu.ac.jp,
WWW home page: http://www.anna.iwate-pu.ac.jp/~twatanab/

Abstract. In this paper, we consider multi-object auctions in which each bidder has a positive reservation value for only one special subset of objects, called a necessary bundle. In the auction, each bidder reports its necessary bundle and its reservation value. The seller solves the assignment problem of objects which maximizes its revenue and decides the winning bidders who can purchase their necessary bundles for their reporting prices. We show that this auction leads to an efficient allocation through Nash equilibria under complete information when the bid-grid size is sufficiently small. We apply our results to spectrum auctions satisfying the conditions that necessary bundles are intervals of discretized radio spectrum. We show that the revenue maximization problem for the seller can be solved in polynomial time for the above auctions. The algorithm also indicates a method to choose an accepted bidder randomly when the revenue maximization problem has multiple optimal solutions. Lastly, we introduce a linear inequality system which characterizes the set of Nash equilibria.

1 Introduction

In this paper, we deal with the multi-object auctions under a specified class of preferences. Multi-object auctions have been used in trading important commodities such as oil leases, treasury securities, furniture and so on. Recently, important applications arise such as pollution rights, airport time slots, assignment of spectrum licenses and delivery routes. As increasing new remarkable applications of multi-object auctions, investigations of them have been developed enormously. These models of multi-object auctions can be characterized by assumptions on bidders' preferences, auction forms and information among the bidders' preferences. Initially, most of the literatures consider the case where objects are identical. The simplest assumption on bidders' preferences is that

each bidder demands only one unit of the objects. Weber's paper [20], a pioneer work of multi-object auctions, considered this case.

In other studies, bidders have positive or negative marginal utilities of items, or complementary or substitution of objects, since it is an important feature to consider the new area of remarkable auctions. Gale [8], Krishna and Rosenthal [10], Levin [12] and Rosenthal and Wang [16] investigated multi-object auctions with synergies or complementarity. Engelbrecht-Wiggans and Kahn [7] and Noussair [14] investigated the case that each bidder has two independent values for objects.

In this paper, we consider multi-object auctions in which each bidder has a positive reservation value for only one special subset of objects, called a necessary bundle. We assume that any object in the set is necessary and sufficient to the bidder. If the bidder misses any object in the set, the bidder has no value for other objects in the set. We also assume that the bidder has no value for objects outside of the set. This is an extreme case of preferences with synergies or complementarities among objects. We assume that the seller knows that each agent has such a preference, and consider the following auction: each bidder reports its necessary bundle and the reservation value for the bundle. The seller solves the assignment problem of the objects which maximizes its revenue and decides the winning bidders who can purchase their necessary bundles for their reporting prices. We call it an auction with necessary bundles. It is applicable to the recent remarkable multi-object auctions as stated above. We study that it leads to an efficient allocation through Nash equilibria under complete information.

Our motivation of this paper is derived from the applications and the sequence of studies listed above, and strongly related to Bikhchandani [3]. The paper [3] analyzed general preferences in which each bidder's reservation value for any subset is given by a real valued function defined on the power set of objects. Many articles on auctions, including multi-object auctions, focus on incomplete information games because auctions can achieve efficient allocations under complete information, but they do not under incomplete information. Bikhchandani [3] also showed that a sealed bid first price auction, in which each object is sold independently, implements Walrasian equilibrium allocations by pure strategy Nash equilibria, when the bid grid-size in the auction is sufficiently small. The sealed bid first price auction, in which each object is sold independently, is also discussed in [4].

Auctions in which each bidder reports its reservation values for any subset of objects may achieve efficient allocations, even if no Walrasian equilibrium exists. Such models are investigated as combinatorial auctions, which have received much attention recently. Vries and Vohra [5] survey articles about combinatorial auctions. Since computing winning bids and allocations is very hard in combinatorial auctions, there are many studies about computational aspects of combinatorial auctions [17]. However, combinatorial auctions have another problem. As the number of objects increases, the number of subsets of the objects increases exponentially. Hence, revealing reservation values for all the subsets is very hard in practice, when the number of objects is large. On our restriction

on preferences such that each bidder has a necessary bundle, our auction form is simple and practically implementable.

In this paper, we discuss the existence of a Nash equilibrium of the multi-objective auctions with necessary bundles. We show that reporting necessary bundles and a minimal price vector in a specified set is a pure strategy Nash equilibrium. This result implies that when the bid grid-size is small enough, a pure strategy Nash equilibrium exists. We apply our results to spectrum auctions analyzed in Krishna and Rosenthal [10] and Rosenthal and Wang [16]. In that auction, necessary bundles are intervals of discretized radio spectrum. We show that the revenue maximization problem for the seller can be solved in polynomial time for the above auctions. The algorithm also indicates a method to choose an accepted bidder randomly when the revenue maximization problem has multiple optimal solutions. Lastly, we introduce a linear inequality system which characterizes the set of Nash equilibria.

This paper is organized as follows. In Section 2, we describe our model and auctions with necessary bundles. In Section 3, we discuss the existence of pure Nash equilibria. In Section 4, we discuss some problems related to spectrum auctions. Section 5 concludes.

2 The Model

2.1 Notation and Definitions

In this paper, we denote the n dimensional vector whose ith component is y_i by $\boldsymbol{y} = (y_1, y_2, \ldots, y_n)$ and the $n-1$ dimensional vector $(y_1, \ldots, y_{i-1}, y_{i+1}, \ldots, y_n)$ by $\boldsymbol{y}_{-i}$. For convenience, the n dimensional vector $(y_1, \ldots, y_{i-1}, z_i, y_{i+1}, \ldots, y_n)$ is denoted by $(z_i, \boldsymbol{y}_{-i})$.

There are n bidders and m indivisible objects. Let $N = \{1, 2, \ldots, n\}$ be the set of bidders, and $M = \{1, 2, \ldots, m\}$ the set of objects. Each subset of objects is called a *bundle*. Each bidder i has a nonnegative reservation value $V_i(S)$ for each bundle $S \subseteq M$. We assume that the reservation value to the empty set is zero for any bidder i, i.e., $V_i(\emptyset) = 0$ for any $i \in N$. We also assume that for any bundle, the reservation value of the seller is equal to zero.

In this paper, we assume that each bidder has a positive reservation value only for one special bundle. This bundle is called the *necessary bundle*. If the bidder misses any object in the bundle, other objects in the bundle are not valuable to the bidder at all. We also assume that the objects in the bundle are also sufficient to the bidder, so the bidder has no value for any object out of the necessary bundle. We denote the necessary bundle of the bidder i by T_i and its value for T_i by $v_i > 0$. Hence, the reservation value $V_i(S)$ is written as follows:

$$V_i(S) = \begin{cases} v_i \ (T \subseteq S), \\ 0 \ \ \text{(otherwise)}. \end{cases}$$

Throughout this paper, we assume that each reservation value is a positive multiple of δ, the *unit of reservation values*, i.e., we assume that $v_i \in \{\delta, 2\delta, 3\delta, \ldots\}$ for any $i \in N$.

2.2 Auctions with Necessary Bundles

We propose the sealed bid simultaneous auctions with necessary bundles. At the beginning of the auction, each bidder $i \in N$ submits a bid (B_i, b_i) where B_i is a bundle and the nonnegative real number b_i is the amount it is willing to pay for the bundle B_i. We assume that there exists a positive integer number I such that each bid is a multiple of $\frac{\delta}{I}$. We write the bid unit as ε, i.e., $\varepsilon = \frac{\delta}{I}$. The set of integer multiples of the bid unit is denoted by $Z_\varepsilon \stackrel{\text{def.}}{=} \{\varepsilon k \mid k \text{ is a nonnegative integer}\}$. Thus, $b_i \in Z_\varepsilon$ for each $i \in N$. In the following, we write a profile of bids $((B_1, b_1), (B_2, b_2), \ldots, (B_n, b_n))$ as $(\boldsymbol{B}, \boldsymbol{b})$ by changing the order of components where $\boldsymbol{B} = (B_1, B_2, \ldots, B_n)$ and $\boldsymbol{b} = (b_1, b_2, \ldots, b_n)$.

The seller solves the following integer programming problem, called the Bundle Assignment Problem (BAP), which maximizes the revenue:

$$\begin{aligned} \mathrm{BAP}(\boldsymbol{B}, \boldsymbol{b}): \quad \text{maximize} \quad & \sum_{i \in N} b_i x_i = \boldsymbol{b} \cdot \boldsymbol{x} \\ \text{subject to} \quad & \sum_{i : B_i \ni j} x_i \leq 1 \quad (\forall j \in M), \\ & x_i \in \{0, 1\} \quad (\forall i \in N), \end{aligned}$$

where $\boldsymbol{x} = (x_1, x_2, \ldots, x_n)$. We denote the set of all the optimal solutions of $\mathrm{BAP}(\boldsymbol{B}, \boldsymbol{b})$ by $\Omega(\boldsymbol{B}, \boldsymbol{b})$. The seller solves the problem $\mathrm{BAP}(\boldsymbol{B}, \boldsymbol{b})$ and obtains an optimal solution $\boldsymbol{x}^*$. If the problem $\mathrm{BAP}(\boldsymbol{B}, \boldsymbol{b})$ has multiple optimal solutions, the seller chooses an optimal solution $\boldsymbol{x}^* \in \Omega(\boldsymbol{B}, \boldsymbol{b})$ at random. The bid by the bidder i is *accepted* if and only if $x_i^* = 1$. Papers [2, 17] discuss some algorithms for solving the problem $\mathrm{BAP}(\boldsymbol{B}, \boldsymbol{b})$.

If the bidder i is accepted, the bidder gets the bundle B_i and pay the price b_i. Given the problem $\mathrm{BAP}(\boldsymbol{B}, \boldsymbol{b})$, we partition the set of bidders by $\mathrm{P}(\boldsymbol{B}, \boldsymbol{b}), \mathrm{Q}(\boldsymbol{B}, \boldsymbol{b})$ and $\mathrm{R}(\boldsymbol{B}, \boldsymbol{b})$ as follows:

$$\begin{aligned} \mathrm{P}(\boldsymbol{B}, \boldsymbol{b}) &\stackrel{\text{def.}}{=} \{i \in N \mid x_i = 1, \ \forall \boldsymbol{x} \in \Omega(\boldsymbol{B}, \boldsymbol{b})\}, \\ \mathrm{R}(\boldsymbol{B}, \boldsymbol{b}) &\stackrel{\text{def.}}{=} \{i \in N \mid x_i = 0, \ \forall \boldsymbol{x} \in \Omega(\boldsymbol{B}, \boldsymbol{b})\}, \\ \mathrm{Q}(\boldsymbol{B}, \boldsymbol{b}) &\stackrel{\text{def.}}{=} N \setminus (\mathrm{P}(\boldsymbol{B}, \boldsymbol{b}) \cup \mathrm{R}(\boldsymbol{B}, \boldsymbol{b})), \end{aligned}$$

where each member in $\mathrm{P}(\boldsymbol{B}, \boldsymbol{b}), \mathrm{Q}(\boldsymbol{B}, \boldsymbol{b})$ and $\mathrm{R}(\boldsymbol{B}, \boldsymbol{b})$ is called a *passed* bidder, a *questionable* bidder, and a *rejected* bidder, respectively.

For the given profile $(\boldsymbol{B}, \boldsymbol{b})$, the expected utility of the bidder i denoted by $U_i(\boldsymbol{B}, \boldsymbol{b})$ is defined as follows:

$$U_i(\boldsymbol{B}, \boldsymbol{b}) \stackrel{\text{def.}}{=} \begin{cases} V_i(B_i) - b_i & (i \in \mathrm{P}(\boldsymbol{B}, \boldsymbol{b})), \\ (V_i(B_i) - b_i) \dfrac{|\{\boldsymbol{x} \in \Omega(\boldsymbol{B}, \boldsymbol{b}) \mid x_i = 1\}|}{|\Omega(\boldsymbol{B}, \boldsymbol{b})|} & (i \in \mathrm{Q}(\boldsymbol{B}, \boldsymbol{b})), \\ 0 & (i \in \mathrm{R}(\boldsymbol{B}, \boldsymbol{b})), \end{cases}$$

where $|\{\boldsymbol{x} \in \Omega(\boldsymbol{B}, \boldsymbol{b}) \mid x_i = 1\}| / |\Omega(\boldsymbol{B}, \boldsymbol{b})|$ is the probability that the bidder i is accepted.

3 Pure Strategy Nash Equilibria

In this section, we discuss the existence of pure strategy Nash equilibria. We say that a profile $(\boldsymbol{B}^*, \boldsymbol{b}^*)$ is a *Nash equilibrium* when $(\boldsymbol{B}^*, \boldsymbol{b}^*)$ satisfies the conditions that for each bidder $i \in N$, $U_i(\boldsymbol{B}^*, \boldsymbol{b}^*) \geq U_i((B_i, \boldsymbol{B}^*_{-i}), (b_i, \boldsymbol{b}^*_{-i}))$ for any bid (B_i, b_i) where $B_i \subseteq M$ and $b_i \in Z_\varepsilon$.

Let us consider the following set:

$$\mathcal{F}_\varepsilon(\boldsymbol{B}, \boldsymbol{v}) \stackrel{\text{def.}}{=} \left\{ \boldsymbol{b} \in Z_\varepsilon^N \;\middle|\; \begin{array}{ll} b_i = v_i & (\forall i \in \mathrm{R}(\boldsymbol{B}, \boldsymbol{v}) \cup \mathrm{Q}(\boldsymbol{B}, \boldsymbol{v})), \\ b_i \leq v_i - 2^n \varepsilon & (\forall i \in \mathrm{P}(\boldsymbol{B}, \boldsymbol{v})), \\ \Omega(\boldsymbol{B}, \boldsymbol{b}) = \Omega(\boldsymbol{B}, \boldsymbol{v}) & \end{array} \right\}$$

which is a subset of bid price vectors satisfying that the set of optimal solutions does not change. Given a subset of bid price vectors $X \subseteq Z_\varepsilon^N$, a vector $\boldsymbol{b} \in X$ is called a *minimal vector in* X if and only if for any $\boldsymbol{b}' \in Z_\varepsilon^N$, the condition $[\boldsymbol{b}' \leq \boldsymbol{b}$ and $\boldsymbol{b}' \neq \boldsymbol{b}]$ implies $\boldsymbol{b}' \notin X$.

Theorem 1. *If $\mathcal{F}_\varepsilon(\boldsymbol{T}, \boldsymbol{v})$ is non-empty, then for any minimal vector $\boldsymbol{b}^*$ in $\mathcal{F}_\varepsilon(\boldsymbol{T}, \boldsymbol{v})$, $(\boldsymbol{T}, \boldsymbol{b}^*)$ is a Nash equilibrium.*

To prove this theorem, we need the following two lemmas whose proofs appear in Appendix.

Lemma 1. *Let $\boldsymbol{B}$ be any bundle vector satisfying $T_i \subseteq B_i$ for any $i \in N$. Suppose $\mathcal{F}_\varepsilon(\boldsymbol{B}, \boldsymbol{v})$ is non-empty and let $\boldsymbol{b}^*$ be a minimal vector of $\mathcal{F}_\varepsilon(\boldsymbol{B}, \boldsymbol{v})$. Then, on a profile of bid $(\boldsymbol{B}, \boldsymbol{b}^*)$, any bidder $i \in N$ can not improve its payoff by changing its price b_i^* to any price $b_i \in Z_\varepsilon$, i.e., $U_i(\boldsymbol{B}, \boldsymbol{b}^*) \geq U_i(\boldsymbol{B}, (b_i, \boldsymbol{b}^*_{-i}))$ for any $b_i \in Z_\varepsilon$.*

Lemma 2. *Let $(\boldsymbol{B}, \boldsymbol{b})$ be any profile of bidding. If the bidder $i \in N$ satisfies $b_i \leq v_i$, then its expected utility of reporting (T_i, b_i) is greater than or equal to reporting (B_i, b_i), i.e., $U_i((T_i, \boldsymbol{B}_{-i}), \boldsymbol{b}) \geq U_i(\boldsymbol{B}, \boldsymbol{b})$.*

Proof of Theorem 1 We have to show that for each bidder $i \in N$, $U_i(\boldsymbol{T}, \boldsymbol{b}^*) \geq U_i((B_i, \boldsymbol{T}_{-i}), (b_i, \boldsymbol{b}^*_{-i}))$ for any bundle B_i and any price b_i. If $T_i \setminus B_i \neq \emptyset$, then $U_i(\boldsymbol{T}, \boldsymbol{b}^*) \geq 0 \geq U_i((B_i, \boldsymbol{T}_{-i}), (b_i, \boldsymbol{b}_{-i}))$ and we have done. If $b_i > v_i$, then we also have $U_i(\boldsymbol{T}, \boldsymbol{b}^*) \geq 0 \geq U_i((B_i, \boldsymbol{T}_{-i}), (b_i, \boldsymbol{b}_{-i}))$.

Suppose $T_i \subseteq B_i$ and $b_i \leq v_i$. Lemma 1 implies $U_i(\boldsymbol{T}, \boldsymbol{b}^*) \geq U_i(\boldsymbol{T}, (b_i, \boldsymbol{b}^*_{-i}))$ and Lemma 2 implies $U_i(\boldsymbol{T}, (b_i, \boldsymbol{b}^*_{-i})) \geq U_i((B_i, \boldsymbol{T}_{-i}), (b_i, \boldsymbol{b}^*_{-i}))$. Hence, we can conclude that $(\boldsymbol{T}, \boldsymbol{b}^*)$ is a Nash equilibrium. ||

The following theorem shows the non-emptiness of $\mathcal{F}_\varepsilon(\boldsymbol{T}, \boldsymbol{v})$.

Theorem 2. *If ε is a sufficiently small positive number, $\mathcal{F}_\varepsilon(\boldsymbol{T}, \boldsymbol{v})$ is non-empty.*

Proof. If $\mathrm{P}(\boldsymbol{T}, \boldsymbol{v}) = \emptyset$, then $\mathcal{F}_\varepsilon(\boldsymbol{T}, \boldsymbol{v}) = \{\boldsymbol{v}\}$ and so $\mathcal{F}_\varepsilon(\boldsymbol{T}, \boldsymbol{v})$ is non-empty. When $\mathrm{P}(\boldsymbol{T}, \boldsymbol{v}) \neq \emptyset$, put

$$d = \min \left\{ (\boldsymbol{v}\boldsymbol{x}' - \boldsymbol{v}\boldsymbol{x}'') \;\middle|\; \begin{array}{l} \text{solutions } \boldsymbol{x}', \boldsymbol{x}'' \text{ are feasible to } \mathrm{BAP}(\boldsymbol{T}, \boldsymbol{v}), \\ (\boldsymbol{v}\boldsymbol{x}' - \boldsymbol{v}\boldsymbol{x}'') > 0 \end{array} \right\}.$$

Now assume that $\varepsilon \leq d/(n2^n+1)$. Let $\boldsymbol{b}$ be the vector defined by,

$$b_i = \begin{cases} v_i & (i \in \mathrm{Q}(\boldsymbol{T},\boldsymbol{v}) \cup \mathrm{R}(\boldsymbol{T},\boldsymbol{v})), \\ v_i - 2^n\varepsilon & (i \in \mathrm{P}(\boldsymbol{T},\boldsymbol{v})). \end{cases}$$

Let $\boldsymbol{x}^*$ be a solution in $\Omega(\boldsymbol{T},\boldsymbol{v})$. Clearly, the equality $\boldsymbol{b}\cdot\boldsymbol{x}^* = \boldsymbol{v}\cdot\boldsymbol{x}^* - 2^n\varepsilon|\mathrm{P}(\boldsymbol{T},\boldsymbol{v})|$ holds.

In the rest of this proof, we show that $\Omega(\boldsymbol{T},\boldsymbol{b}) = \Omega(\boldsymbol{T},\boldsymbol{v})$. If $\boldsymbol{x}'$ is a solution satisfying $\boldsymbol{v}\cdot\boldsymbol{x}' < \boldsymbol{v}\cdot\boldsymbol{x}^*$, then

$$\begin{aligned} \boldsymbol{b}\cdot\boldsymbol{x}' &\leq \boldsymbol{v}\cdot\boldsymbol{x}' \leq \boldsymbol{v}\cdot\boldsymbol{x}^* - d \leq \boldsymbol{v}\cdot\boldsymbol{x}^* - (n2^n+1)\varepsilon \leq \boldsymbol{v}\cdot\boldsymbol{x}^* - (|\mathrm{P}(\boldsymbol{T},\boldsymbol{v})|2^n+1)\varepsilon \\ &< \boldsymbol{v}\cdot\boldsymbol{x}^* - 2^n\varepsilon|\mathrm{P}(\boldsymbol{T},\boldsymbol{v})| = \boldsymbol{b}\cdot\boldsymbol{x}^*, \end{aligned}$$

and so $\boldsymbol{b}\cdot\boldsymbol{x}' < \boldsymbol{b}\cdot\boldsymbol{x}^*$. If $\boldsymbol{v}\cdot\boldsymbol{x}' = \boldsymbol{v}\cdot\boldsymbol{x}^*$, then $\boldsymbol{x}' \in \Omega(\boldsymbol{T},\boldsymbol{v})$ and so

$$\boldsymbol{b}\cdot\boldsymbol{x}' = \boldsymbol{v}\cdot\boldsymbol{x}' - 2^n\varepsilon|\mathrm{P}(\boldsymbol{T},\boldsymbol{v})| = \boldsymbol{v}\cdot\boldsymbol{x}^* - 2^n\varepsilon|\mathrm{P}(\boldsymbol{T},\boldsymbol{v})| = \boldsymbol{b}\cdot\boldsymbol{x}^*.$$

Thus, we have $\Omega(\boldsymbol{T},\boldsymbol{b}) = \Omega(\boldsymbol{T},\boldsymbol{v})$.

From the above, $\boldsymbol{b} \in \mathcal{F}_\varepsilon(\boldsymbol{T},\boldsymbol{v})$ and so $\mathcal{F}_\varepsilon(\boldsymbol{T},\boldsymbol{v})$ is non-empty.

For characterizing Nash equilibria in practice, we need to describe the set $\mathcal{F}_\varepsilon(\boldsymbol{B},\boldsymbol{v})$ in a tractable form. The following lemma gives an alternative description of $\mathcal{F}_\varepsilon(\boldsymbol{B},\boldsymbol{v})$, which enables us to separate conditions related to the discretization with bid unit ε. For any $\mathcal{G} \subseteq \mathrm{R}^N$, $\mathrm{int}\mathcal{G}$ denotes the set of relative interior points in $\mathcal{G}$. When $\mathcal{G}$ is a singleton, we define that $\mathrm{int}\mathcal{G} = \mathcal{G}$.

Lemma 3. *Let $\overline{\mathcal{F}}(\boldsymbol{B},\boldsymbol{v})$ be the set defined by*

$$\overline{\mathcal{F}}(\boldsymbol{B},\boldsymbol{v}) \stackrel{\mathrm{def.}}{=} \left\{ \boldsymbol{b} \in \mathrm{R}^N \;\middle|\; \begin{array}{l} b_i = v_i \;\; (\forall i \in \mathrm{R}(\boldsymbol{B},\boldsymbol{v}) \cup \mathrm{Q}(\boldsymbol{B},\boldsymbol{v})), \\ b_i \leq v_i \;\; (\forall i \in \mathrm{P}(\boldsymbol{B},\boldsymbol{v})), \\ \Omega(\boldsymbol{B},\boldsymbol{b}) \supseteq \Omega(\boldsymbol{B},\boldsymbol{v}) \end{array} \right\}.$$

Then, $\mathcal{F}_\varepsilon(\boldsymbol{B},\boldsymbol{v}) = \mathrm{int}\overline{\mathcal{F}}(\boldsymbol{B},\boldsymbol{v}) \cap \{\boldsymbol{b} \in Z_\varepsilon^N \mid b_i \leq v_i - 2^n\varepsilon \;(\forall i \in \mathrm{P}(\boldsymbol{B},\boldsymbol{v}))\}$.

The above lemma gives an idea for describing the set of Nash equilibria in a tractable form. We apply the above lemma to an application setting later.

4 Spectrum Auctions

In this section, we consider a spectrum auction. An auctioneer wants to sell licenses for radio spectrum $M = \{1,2,\cdots,m\}$. Each spectrum license $j(\geq 2)$ is adjacent to $j-1$ on the right. Kirishna and Rosenthal [10] and Rosenthal and Wang [16] analyzed models in which each object has two neighboring objects on the right and the left, and "global" bidders want to purchase the object together with its neighbor on the left or the right. We consider the case of the preferences in which each bidder i requires any spectrum license j satisfying $\ell_i \leq j \leq h_i$ but no spectrum license outside of it. This setting is applied to our

model, auctions with necessary bundles, where agent i's necessary bundle is an interval $T_i = \{j \in M \mid \ell_i \leq j \leq h_i\}$.

In this case, we show that the related bundle assignment problem (BAP) is polynomially solvable. We also discuss a method to choose an optimal solution randomly, when BAP has multiple optimal solutions. Lastly, we describe a linear inequality system which includes the set of Nash equilibria as the set of feasible minimal vectors.

4.1 Bundle Assignment Problem and Longest Path Problem

Here we denote BAP by $\max\{\boldsymbol{b} \cdot \boldsymbol{x} \mid A\boldsymbol{x} \leq \mathbf{1},\ \boldsymbol{x} \in \{0,1\}^N\}$. Then the coefficient matrix $A \in \mathrm{R}^{M \times N}$ belongs to the class of consecutive one matrices, where each column vector $(a_{1i}, a_{2i}, \ldots, a_{mi})^\top$ satisfies the condition that there exists a pair of row indices (ℓ_i, h_i) satisfying that

$$a_{ji} = \begin{cases} 0\ (j < \ell_i), \\ 1\ (\ell_i \leq j \leq h_i), \\ 0\ (h_i < j). \end{cases}$$

It is well-known that the consecutive one matrices are totally unimodular, i.e., the determinant of each square submatrix is either 0, -1 or 1 (see [18] for example). And so, the linear relaxation problem of BAP, $\max\{\boldsymbol{b} \cdot \boldsymbol{x} \mid A\boldsymbol{x} \leq \mathbf{1},\ \mathbf{0} \leq \boldsymbol{x} \leq \mathbf{1}\}$, has a 0-1 valued optimal extreme point solution for any vector $\boldsymbol{b}$. Thus, when we solve the above linear programming problem by polynomial time method, we can obtain an optimal solution to BAP in polynomial time. In the following, we show that the above problem is essentially equivalent to the longest path problem and we can solve the problem by CPM (critical path method) for PERT (Program Evaluation and Review Technique).

First, we construct a directed graph $G = (V, A)$ with the vertex set $V = \{0\} \cup M = \{0, 1, 2, \ldots, m\}$ and the directed arc set $E = E_1 \cup E_2$ defined by

$$E_1 \stackrel{\text{def.}}{=} \{(0,1), (1,2), \ldots, (m-1, m)\} \quad \text{and} \quad E_2 \stackrel{\text{def.}}{=} \{(\ell_i - 1, h_i) \mid i \in N\}.$$

For any directed path P in G, we define the corresponding set of bidders $N(P)$ by

$$N(P) \stackrel{\text{def.}}{=} \{i \in N \mid \text{ the arc } (\ell_i - 1, h_i) \text{ is contained in } P\}.$$

Then, for any directed path P from 0 to m in G, $N(P)$ satisfies the property that bundles in $\{T_i \mid i \in N(P)\}$ are mutually disjoint. Conversely, for any set of mutually disjoint bundles $\{T_{i_1}, T_{i_2}, \ldots, T_{i_k}\}$, there exists a path P' from 0 to m in G satisfying that $N(P') = \{i_1, i_2, \ldots, i_k\}$. For each arc $e \in E$, we define its arc weight $w(e)$ by;

$$w_{\boldsymbol{b}}(e) \stackrel{\text{def.}}{=} \begin{cases} b_i\ (e = (\ell_i - 1, h_i) \in E_2), \\ 0\ \ (e \in E_1 \setminus E_2). \end{cases}$$

Since BAP finds a set of mutually disjoint bundles which maximizes the total revenue, the problem is essentially equivalent to the problem for finding a longest

directed path from 0 to m in G. The critical path method, which is equivalent to the ordinary dynamic programming technique, finds a longest path from 0 to m in G in $O(n+m)$ time (see [1, 6, 11] for example. The method also finds the longest distance from 0 to j for every vertex $j \in V$ without increasing the time complexity.

4.2 Random Selection from Multiple Optimal Solutions

When BAP has multiple optimal solutions, seller needs to choose an optimal solution at random. Here we describe a polynomial time method for choosing a longest path in G randomly. In the following, we denote the longest distance from vertex 0 to vertex j in G by d_j for each vertex $j \in V$. The *admissible graph* is a directed subgraph $G^* = (V, E^*)$ of G where $E^* = \{e' = (j', j'') \in E \mid d_{j''} - d_{j'} = w(e')\}$. Then it is easy to show that a directed path P from 0 to m in G is a longest path if and only if P is contained in the admissible graph G^*. Thus, the seller only needs to choose a path from 0 to m in G^* at random. For each vertex j in G^*, α_j denotes the number of directed paths in G^* from j to m. We can calculate α_j for each vertex $j \in V$ by solving the following recursive equality system;

$$\begin{aligned} \alpha_m &= 1, \\ \alpha_j &= \sum_{(j,j') \in E^*} \alpha_{j'} \quad (j \in V \setminus \{m\}). \end{aligned}$$

Since each arc (j, j') satisfies $j < j'$, we can obtain the vector $(\alpha_0, \alpha_1, \ldots, \alpha_m)$ in decreasing order of indices. The above procedure requires $O(n+m)$ time. Then, seller can choose a path in G^* randomly, as follows.

Randomized Path Selection

Step 0: Set $P := (0)$ and $j := 0$.
Step 1: If $j = m$, then output the path corresponding to P and stop.
Step 2: Set $J := \{j' \in V \mid (j, j') \in E^*\}$ and choose a vertex $j^* \in J$ with the probability that $\Pr[j^* = j'] \stackrel{\text{def.}}{=} \alpha_{j'} / \sum_{j'' \in J} \alpha_{j''}$. Add the vertex j^* to the tail of the vertex sequence P. Set $j := j^*$ and go to Step 1.

If we can choose a vertex j^* in Step 2 in a constant time, then the time complexity of the above algorithm is bounded by $O(m+n)$ time.

4.3 Nash Equilibria

Here we show an inequality system for describing the region $\overline{\mathcal{F}}(\boldsymbol{B}, \boldsymbol{v})$ of the spectrum auction defined above.

First, we consider the sets of passed, questionable and rejected bidders. As described in the previous subsection, we can calculate α_j , the number of directed paths in G^* from j to m, in linear time. Put $E_3 \stackrel{\text{def.}}{=} \{(j, j') \in E^* \mid \alpha_{j'} > 0\}$. Then,

for each arc $e \in E_3$, there exists at least one path from 0 to m in G^* including e, it is because G^* contains a path from 0 to j for each vertex $j \in V$. It implies that the set of rejected bidders is equivalent to $\{i \in N \mid (\ell_i - 1, h_i) \notin E_3\}$. An arc e is called 0-m *cut-arc* of (V, E_3), if every 0-m path in (V, E_3) contains the arc e. It is clear that each arc $(\ell_i - 1, h_i) \in E_3$ corresponds to a passed bidder if and only if the arc is a 0-m cut-arc of (V, E_3). The ordinary depth-first search on directed graph finds all the 0-m cut-arcs in linear time [19]. From the above, we can construct the sets of passed, questionable and rejected bidders in $\mathrm{O}(n+m)$ time.

For any subset of bidders N', we denote the arc subset $\{(\ell_i - 1, h_i) \mid i \in N'\}$ by $E(N')$. In the following, we denote the set of passed bidders by $\mathrm{P}(\boldsymbol{B}, \boldsymbol{v}) = \{p_1, p_2, \ldots, p_s\}$ where $\ell_{p_1} < \ell_{p_2} < \cdots < \ell_{p_s}$. The definition of 0-$m$ cut-arc implies that $\ell_{p_1} \leq h_{p_1} < \ell_{p_2} \leq h_{p_2} < \cdots < \ell_{p_s} \leq h_{p_s}$. A subset N' of passed bidders is called an *interval of passed bidders*, when there exists a pair of indices (k, k') satisfying that $N' = \{p_k, p_{k+1}, \ldots, p_{k'}\}$. The family of all the intervals of passed bidders is denoted by $\mathcal{I}(\boldsymbol{B}, \boldsymbol{v})$. For each interval of passed bidders $N' \in \mathcal{I}(\boldsymbol{B}, \boldsymbol{v})$, we choose a longest 0-$m$ path in the graph $(V, E \setminus E(N'))$ with arc weight $w_{\boldsymbol{v}}$ if it exists, and denote the path by $P(N')$. In case of non-existence, we set $P(N') = \emptyset$. We choose a longest path in the graph G with respect to $w_{\boldsymbol{v}}$ and denote the path by P^*. Then we have the following lemma.

Lemma 4.

$$\overline{\mathcal{F}}(\boldsymbol{B}, \boldsymbol{v}) = \left\{ \boldsymbol{b} \in \mathrm{R}^N \;\middle|\; \begin{array}{ll} b_i = v_i & (\forall i \in \mathrm{R}(\boldsymbol{B}, \boldsymbol{v}) \cup \mathrm{Q}(\boldsymbol{B}, \boldsymbol{v})), \\ b_i \leq v_i & (\forall i \in \mathrm{P}(\boldsymbol{B}, \boldsymbol{v})), \\ \displaystyle\sum_{i \in N(P^*)} b_i \geq \sum_{i \in N(P(N'))} b_i & (\forall N' \in \mathcal{I}(\boldsymbol{B}, \boldsymbol{v})) \end{array} \right\}.$$

Proof.

$$\text{Put } \mathcal{F}' = \left\{ \boldsymbol{b} \in \mathrm{R}^N \;\middle|\; \begin{array}{ll} b_i = v_i & (\forall i \in \mathrm{R}(\boldsymbol{B}, \boldsymbol{v}) \cup \mathrm{Q}(\boldsymbol{B}, \boldsymbol{v})), \\ b_i \leq v_i & (\forall i \in \mathrm{P}(\boldsymbol{B}, \boldsymbol{v})), \\ \displaystyle\sum_{i \in N(P^*)} b_i \geq \sum_{i \in N(P(N'))} b_i & (\forall N' \in \mathcal{I}(\boldsymbol{B}, \boldsymbol{v})) \end{array} \right\}.$$

First, we show that $\overline{\mathcal{F}}(\boldsymbol{B}, \boldsymbol{v}) \subseteq \mathcal{F}'$. For any $\boldsymbol{b}^* \in \overline{\mathcal{F}}(\boldsymbol{B}, \boldsymbol{v})$, $\boldsymbol{b}^*$ satisfies the inequality $\sum_{i \in N(P^*)} b_i^* \geq \sum_{i \in N(P')} b_i^*$ for each 0-m path P' in G, since $\Omega(\boldsymbol{B}, \boldsymbol{b}) \supseteq \Omega(\boldsymbol{B}, \boldsymbol{v})$. Thus, we have $\boldsymbol{b}^* \in \mathcal{F}'$.

Next, we show the converse. Let $\boldsymbol{b}^*$ be a vector in $\mathcal{F}'$. Assume on the contrary that $\boldsymbol{b}^* \notin \overline{\mathcal{F}}(\boldsymbol{B}, \boldsymbol{v})$. Then, there exists a path P' in G satisfying that $\sum_{i \in N(P^*)} b_i^* < \sum_{i \in N(P')} b_i^*$. Assume that P' is a path satisfying the inequality $\sum_{i \in N(P^*)} b_i^* < \sum_{i \in N(P')} b_i^*$ which minimizes the cardinality of $N' = N(P') \cap \mathrm{P}(\boldsymbol{B}, \boldsymbol{v})$. If N' is an interval of passed bidders, then $\sum_{i \in N(P')} b_i^* \leq \sum_{i \in N(P(N'))} b_i^* \leq \sum_{i \in N(P^*)} b_i^*$ and it is a contradiction. We only need to consider the case that N' is not an interval of passed bidders. Put $N' = \{p_{i_1}, p_{i_2}, \ldots, p_{i_{s'}}\}$ where $i_1 < i_2 < \cdots < i_{s'}$. Since N' is not an interval of passed bidders, there exists a passed bidder $p_{i^*} \notin N'$ satisfying that $i_1 < i^* < i_{s'}$. We denote the arc

in E_2 corresponding to the bidder p_{i^*} by $(\ell^* - 1, h^*)$. We partition the path P' to two paths P'_1 and P'_2 satisfying that P'_1 is a subpath of P' from 0 to h^* and P'_2 is a subpath of P' from h^* to m. We also partition the path P^* at vertex h^* and denote the path from 0 to h^* by P^*_1 and the path from h^* to m by P^*_2. Then, either of the following inequalities holds;

$$\sum_{i \in N(P^*)} b^*_i < \sum_{i \in P'_1} b^*_i + \sum_{i \in P^*_2} b^*_i, \quad \sum_{i \in N(P^*)} b^*_i < \sum_{i \in P^*_1} b^*_i + \sum_{i \in P'_2} b^*_i.$$

Thus, either (P^*_1, P'_2) or (P'_1, P^*_2) is longer than P^* and it contradicts with the minimality of the cardinality of $N' = N(P') \cap \mathrm{P}(\boldsymbol{B}, \boldsymbol{v})$ of P'.

The above lemma implies that we can characterize the set of Nash equilibria as a set of minimal interior points in the polyhedral set defined by a system of linear inequalities consists of $\mathrm{O}(\min\{n^2, m^2\} + n + m)$ inequalities.

5 Conclusion

In this paper, we consider multi-object auctions in which each bidder has a positive reservation value for only one special subset of objects, called a necessary bundle. We showed that it leads to an efficient allocation through Nash equilibria under complete information. We also discussed the spectrum auctions analyzed in Krishna and Rosenthal [10] and Rosenthal and Wang [16]. In that auction, necessary bundles are intervals of discretized radio spectrum. We showed that the related bundle assignment problem (BAP) is polynomially solvable by using critical path method for PERT. We also discuss a method to choose an optimal solution randomly, when BAP has multiple optimal solutions. Lastly, we describe a polynomial size linear inequality system which includes the set of Nash equilibria as the set of feasible minimal interior vectors.

Appendix

Proof of Lemma 1.

We have $P(\boldsymbol{B}, \boldsymbol{v}) = P(\boldsymbol{B}, \boldsymbol{b}^*)$, $Q(\boldsymbol{B}, \boldsymbol{v}) = Q(\boldsymbol{B}, \boldsymbol{b}^*)$, and $R(\boldsymbol{B}, \boldsymbol{v}) = R(\boldsymbol{B}, \boldsymbol{b}^*)$ from the definition of a minimal vector. We also remark that any feasible solution x of $\mathrm{BAP}(\boldsymbol{B}, \boldsymbol{b}^*)$ is also feasible to $\mathrm{BAP}(\boldsymbol{B}, \boldsymbol{b})$ for any $\boldsymbol{b}$, so we denote that it is feasible for $\mathrm{BAP}(\boldsymbol{B})$.

(case 1) First, we consider the case that i is a rejected bidder of $\mathrm{BAP}(\boldsymbol{B}, \boldsymbol{v})$, i.e., $i \in \mathrm{R}(\boldsymbol{B}, \boldsymbol{v})$. From the definition of $\mathcal{F}_\varepsilon(\boldsymbol{B}, \boldsymbol{v})$, $b^*_i = v_i$. If the bidder $i \in \mathrm{R}(\boldsymbol{B}, \boldsymbol{v})$ submit a bid price b_i satisfying $b_i < b^*_i = v_i$ instead of b^*_i, the bidder i will not be accepted and so the payoff $U_i(\boldsymbol{B}, (b_i, \boldsymbol{b}^*_{-i}))$ is equal to 0. Let us consider the case that the bidder $i \in \mathrm{R}(\boldsymbol{B}, \boldsymbol{v})$ submit a bid price b_i satisfying $b_i > b^*_i = v_i$. If $i \in \mathrm{R}(\boldsymbol{B}, (b_i, \boldsymbol{b}^*_{-i}))$, $U_i(\boldsymbol{B}, (b_i, \boldsymbol{b}^*_{-i}))$ is also equal to zero. If $i \in \mathrm{P}(\boldsymbol{B}, (b_i, \boldsymbol{b}^*_{-i})) \cup \mathrm{Q}(\boldsymbol{B}, (b_i, \boldsymbol{b}^*_{-i}))$, then $U_i(\boldsymbol{B}, (b_i, \boldsymbol{b}^*_{-i})) = (v_i - b_i) | \{\boldsymbol{x} \in \Omega(\boldsymbol{B}, (b_i, \boldsymbol{b}^*_{-i})) \mid x_i =$

$1\}|/|\Omega(\boldsymbol{B},(b_i,\boldsymbol{b}^*_{-i}))| < 0 = U_i(\boldsymbol{B},\boldsymbol{b}^*)$. From the above, if i is rejected bidder, then we have $U_i(\boldsymbol{B},\boldsymbol{b}^*) \geq U_i(\boldsymbol{B},(b_i,\boldsymbol{b}^*_{-i}))$.

(case 2) Next, we consider a questionable bidder $i \in \mathrm{Q}(\boldsymbol{B},\boldsymbol{v})$. From the definition of $\mathcal{F}_\varepsilon(\boldsymbol{B},\boldsymbol{v})$, $b^*_i = v_i$. If $b_i > v_i$, then $\Omega(\boldsymbol{B},(b_i,\boldsymbol{b}^*_{-i})) = \{\boldsymbol{x} \in \Omega(\boldsymbol{B},\boldsymbol{b}^*) \mid x_i = 1\}$ and $i \in \mathrm{P}(\boldsymbol{B},(b_i,\boldsymbol{b}^*_{-i}))$. Hence, we have $U_i(\boldsymbol{B},(b_i,\boldsymbol{b}^*_{-i})) = v_i - b_i < 0 \leq U_i(\boldsymbol{B},\boldsymbol{b}^*)$. Now consider the case that $b_i < v_i$, then $\Omega(\boldsymbol{B},(b_i,\boldsymbol{b}^*_{-i})) = \{\boldsymbol{x} \in \Omega(\boldsymbol{B},\boldsymbol{b}^*) \mid x_i = 0\}$ and $i \in \mathrm{R}(\boldsymbol{B},(b_i,\boldsymbol{b}^*_{-i}))$. Hence, we have $U_i(\boldsymbol{B},(b_i,\boldsymbol{b}^*_{-i})) = 0 \leq U_i(\boldsymbol{B},\boldsymbol{b}^*)$. From the above, if i is a questionable bidder, $U_i(\boldsymbol{B},\boldsymbol{b}^*) \geq U_i(\boldsymbol{B},(b_i,\boldsymbol{b}^*_{-i}))$.

(case 3) Lastly, we consider the case that $i \in \mathrm{P}(\boldsymbol{B},\boldsymbol{v})$. If $b_i > b^*_i$, then $\Omega(\boldsymbol{B},(b_i,\boldsymbol{b}^*_{-i})) = \Omega(\boldsymbol{B},\boldsymbol{b}^*)$. This implies $i \in \mathrm{P}(\boldsymbol{B},(b_i,\boldsymbol{b}^*_{-i}))$ and so $U_i(\boldsymbol{B},(b_i,\boldsymbol{b}^*_{-i}))_i = v_i - b_i < v_i - b^*_i = U_i(\boldsymbol{B},\boldsymbol{b}^*)$. Now consider the case that $b_i < b^*_i$. The minimality of $\boldsymbol{b}^*$ and the definition of $\mathcal{F}_\varepsilon(\boldsymbol{B},\boldsymbol{v})$ implies that $\Omega(\boldsymbol{B},\boldsymbol{b}^*) \neq \Omega(\boldsymbol{B},(b_i,\boldsymbol{b}^*_{-i}))$. If $i \in \mathrm{R}(\boldsymbol{B},(b_i,\boldsymbol{b}^*_{-i}))$, then $U_i(\boldsymbol{B},(b_i,\boldsymbol{b}^*_{-i})) = 0 \leq U_i(\boldsymbol{B},\boldsymbol{b}^*)$. In the rest of this proof, we consider the remained case that $i \in \mathrm{P}(\boldsymbol{B},\boldsymbol{v}) = \mathrm{P}(\boldsymbol{B},\boldsymbol{b}^*)$, $b_i < b^*_i$, and $i \in \mathrm{P}(\boldsymbol{B},(b_i,\boldsymbol{b}^*_{-i})) \cup \mathrm{Q}(\boldsymbol{B},(b_i,\boldsymbol{b}^*_{-i}))$.

(3-1) First, we show that $\Omega(\boldsymbol{B},\boldsymbol{b}^*) = \{\boldsymbol{x} \in \Omega(\boldsymbol{B},(b_i,\boldsymbol{b}^*_{-i})) \mid x_i = 1\}$. Since $i \in \mathrm{P}(\boldsymbol{B},(b_i,\boldsymbol{b}^*_{-i})) \cup \mathrm{Q}(\boldsymbol{B},(b_i,\boldsymbol{b}^*_{-i}))$, there exists an optimal solution $\boldsymbol{x}' \in \Omega(\boldsymbol{B},(b_i,\boldsymbol{b}^*_{-i}))$ satisfying that $x'_i = 1$. For any optimal solution $\boldsymbol{x}^* \in \Omega(\boldsymbol{B},\boldsymbol{b}^*)$,

$$(b_i,\boldsymbol{b}^*_{-i})\boldsymbol{x}^* = \boldsymbol{b}^*\boldsymbol{x}^* - (b^*_i - b_i) \geq \boldsymbol{b}^*\boldsymbol{x}' - (b^*_i - b_i) = (b_i,\boldsymbol{b}^*_{-i})\boldsymbol{x}',$$

and so $\boldsymbol{x}^*$ is optimal to $\mathrm{BAP}(\boldsymbol{B},(b_i,\boldsymbol{b}^*_{-i}))$. Since $i \in \mathrm{P}(\boldsymbol{B},\boldsymbol{b}^*)$, $\boldsymbol{x}^* \in \Omega(\boldsymbol{B},(b_i,\boldsymbol{b}^*_{-i}))$ implies that $\Omega(\boldsymbol{B},\boldsymbol{b}^*) \subseteq \{\boldsymbol{x} \in \Omega(\boldsymbol{B},(b_i,\boldsymbol{b}^*_{-i})) \mid x_i = 1\}$. Conversely, for any $\boldsymbol{x}' \in \{\boldsymbol{x} \in \Omega(\boldsymbol{B},(b_i,\boldsymbol{b}^*_{-i})) \mid x_i = 1\}$,

$$b^*\boldsymbol{x}' = (b_i,\boldsymbol{b}^*_{-i})\boldsymbol{x}' + (b^*_i - b_i) \geq (b_i,\boldsymbol{b}^*_{-i})x^* + (b^*_i - b_i) = \boldsymbol{b}^*\boldsymbol{x}^*,$$

and so $\boldsymbol{x}'$ is also optimal to $\mathrm{BAP}(\boldsymbol{B},\boldsymbol{b}^*)$. It implies $\Omega(\boldsymbol{B},\boldsymbol{b}^*) \supseteq \{\boldsymbol{x} \in \Omega(\boldsymbol{B},(b_i,\boldsymbol{b}^*_{-i})) \mid x_i = 1\}$ and consequently, $\Omega(\boldsymbol{B},\boldsymbol{b}^*) = \{\boldsymbol{x} \in \Omega(\boldsymbol{B},(b_i,\boldsymbol{b}^*_{-i})) \mid x_i = 1\}$. The property $\Omega(\boldsymbol{B},\boldsymbol{b}^*) = \{\boldsymbol{x} \in \Omega(\boldsymbol{B},(b_i,\boldsymbol{b}^*_{-i})) \mid x_i = 1\}$ implies $i \notin \mathrm{P}(\boldsymbol{B},(b_i,\boldsymbol{b}^*_{-i}))$. It is because, if $i \in P(\boldsymbol{B},(b_i,\boldsymbol{b}^*_{-i}))$, then $\Omega(\boldsymbol{B},(b_i,\boldsymbol{b}^*_{-i})) = \{\boldsymbol{x} \in \Omega(\boldsymbol{B},(b_i,\boldsymbol{b}^*_{-i})) \mid x_i = 1\}$ and so $\Omega(\boldsymbol{B},(b_i,\boldsymbol{b}^*_{-i})) = \Omega(\boldsymbol{B},\boldsymbol{b}^*)$. It is contradicts with the minimality of $\boldsymbol{b}^*$. Thus, we only need to consider the case that $i \in \mathrm{Q}(\boldsymbol{B},(b_i,\boldsymbol{b}^*_{-i}))$.

(3-2) We show that $b_i = b^*_i - \varepsilon$. Let $\overline{\boldsymbol{b}}$ be the vector obtained from $\boldsymbol{b}^*$ by substituting b^*_i by $b^*_i - \varepsilon$. The definition of $\mathcal{F}_\varepsilon(\boldsymbol{B},\boldsymbol{v})$ and the minimality of $\boldsymbol{b}^*$ implies that $\Omega(\boldsymbol{B},\overline{\boldsymbol{b}}) \neq \Omega(\boldsymbol{B},\boldsymbol{b}^*)$. In the following, we show that the assumption that $b_i < b^*_i - \varepsilon$ implies $\Omega(\boldsymbol{B},\overline{\boldsymbol{b}}) = \Omega(\boldsymbol{B},\boldsymbol{b}^*)$. Let $\boldsymbol{x}^*$ be a solution in $\Omega(\boldsymbol{B},\boldsymbol{b}^*)$ and $\boldsymbol{x}$ be any feasible solution of $\mathrm{BAP}(\boldsymbol{B})$. If $x_i = 0$, then the property $\boldsymbol{x}^* \in \Omega(\boldsymbol{B},\boldsymbol{b}^*) \subseteq \Omega(\boldsymbol{B},(b_i,\boldsymbol{b}^*_{-i}))$ implies that

$$\overline{\boldsymbol{b}}\boldsymbol{x} = (b_i,\boldsymbol{b}^*_{-i})\boldsymbol{x} \leq (b_i,\boldsymbol{b}^*_{-i})\boldsymbol{x}^* \leq \overline{\boldsymbol{b}}\boldsymbol{x}^*.$$

If $x_i = 1$, then

$$\overline{\boldsymbol{b}}\boldsymbol{x} = \boldsymbol{b}^*\boldsymbol{x} - \varepsilon \leq \boldsymbol{b}^*\boldsymbol{x}^* - \varepsilon = \overline{\boldsymbol{b}}\boldsymbol{x}^*.$$

From the above inequalities, $\boldsymbol{x}^*$ is optimal to BAP($\boldsymbol{B}$,$\overline{\boldsymbol{b}}$) and so $\Omega(\boldsymbol{B},\boldsymbol{b}^*) \subseteq \Omega(\boldsymbol{B},\overline{\boldsymbol{b}})$. Next, we prove that $\Omega(\boldsymbol{B},\boldsymbol{b}^*) \supseteq \Omega(\boldsymbol{B},\overline{\boldsymbol{b}})$ by showing that for any feasible solution $\boldsymbol{x}$ of BAP($\boldsymbol{B}$), $\boldsymbol{x} \notin \Omega(\boldsymbol{B},\boldsymbol{b}^*)$ implies $\boldsymbol{x} \notin \Omega(\boldsymbol{B},\overline{\boldsymbol{b}})$. When $x_i = 0$, $\boldsymbol{x}$ satisfies that

$$\overline{\boldsymbol{b}}\boldsymbol{x} = (b_i, \boldsymbol{b}^*_{-i})\boldsymbol{x} \leq (b_i, \boldsymbol{b}^*_{-i})\boldsymbol{x}^* = \overline{\boldsymbol{b}}\boldsymbol{x}^* - \varepsilon < \overline{\boldsymbol{b}}\boldsymbol{x}^*$$

and so $\boldsymbol{x}$ is not optimal to BAP($\boldsymbol{B}$,$\overline{\boldsymbol{b}}$). If $\boldsymbol{x}$ satisfies that $x_i = 1$,

$$\overline{\boldsymbol{b}}\boldsymbol{x} = \boldsymbol{b}^*\boldsymbol{x} - \varepsilon < \boldsymbol{b}^*\boldsymbol{x}^* - \varepsilon = \overline{\boldsymbol{b}}\boldsymbol{x}^*,$$

and so $\boldsymbol{x}$ is not optimal to BAP($\boldsymbol{B}$,$\overline{\boldsymbol{b}}$). It implies that $\Omega(\boldsymbol{B},\boldsymbol{b}^*) \supseteq \Omega(\boldsymbol{B},\overline{\boldsymbol{b}})$. From the above $\Omega(\boldsymbol{B},\boldsymbol{b}^*) = \Omega(\boldsymbol{B},\overline{\boldsymbol{b}})$ and it is a contradiction.

(3-3) From the above discussions, we have that $b_i = b_i^* - \varepsilon$ and $i \in Q(\boldsymbol{B},\boldsymbol{b})$. Then,

$$\begin{aligned}
U_i(\boldsymbol{B},(b_i,\boldsymbol{b}^*_{-i})) &= (v_i - b_i)\frac{\left|\{\boldsymbol{x} \in \Omega(\boldsymbol{B},(b_i,\boldsymbol{b}^*_{-i})) \mid x_i = 1\}\right|}{\left|\Omega(\boldsymbol{B},(b_i,\boldsymbol{b}^*_{-i}))\right|} \\
&= (v_i - b_i)\frac{|\Omega(\boldsymbol{B},\boldsymbol{b}^*)|}{\left|\Omega(\boldsymbol{B},(b_i,\boldsymbol{b}^*_{-i}))\right|} = (v_i - (b_i^* - \varepsilon))\frac{|\Omega(\boldsymbol{B},\boldsymbol{b}^*)|}{\left|\Omega(\boldsymbol{B},(b_i,\boldsymbol{b}^*_{-i}))\right|} \\
&= (v_i - b_i^*) - (v_i - b_i^*)\frac{\left|\Omega(\boldsymbol{B},(b_i,\boldsymbol{b}^*_{-i}))\right| - |\Omega(\boldsymbol{B},\boldsymbol{b}^*)|}{\left|\Omega(\boldsymbol{B},(b_i,\boldsymbol{b}^*_{-i}))\right|} + \varepsilon\frac{|\Omega(\boldsymbol{B},\boldsymbol{b}^*)|}{\left|\Omega(\boldsymbol{B},(b_i,\boldsymbol{b}^*_{-i}))\right|} \\
&\leq U_i(\boldsymbol{B},\boldsymbol{b}^*) - (v_i - b_i^*)/2^n + \varepsilon \leq U_i(\boldsymbol{B},\boldsymbol{b}^*) - \varepsilon + \varepsilon = U_i(\boldsymbol{B},\boldsymbol{b}^*),
\end{aligned}$$

since the definition of $\mathcal{F}_\varepsilon$ implies that $b_i^* \leq v_i - 2^n\varepsilon$.

From the above, b_i^* is a best reply with respect to $\boldsymbol{b}^*$ and so we have that for each bidder i, $U_i(\boldsymbol{B},\boldsymbol{b}^*) \geq U_i(\boldsymbol{B},(b_i,\boldsymbol{b}^*_{-i}))$. ||

Proof of Lemma 2.

Clearly, $U_i((T_i,\boldsymbol{B}_{-i}),\boldsymbol{b})$ is nonnegative. If $T_i \setminus B_i \neq \emptyset$, then $U_i(\boldsymbol{B},\boldsymbol{b}) \leq 0 \leq U_i((T_i,\boldsymbol{B}_{-i}),\boldsymbol{b})$. Thus, we only need to consider the case $T_i \subseteq B_i$. In this case, each feasible solution for BAP($\boldsymbol{B}$,$\boldsymbol{b}$) is also feasible to BAP($(T_i,\boldsymbol{B}_{-i})$,$\boldsymbol{b}$).

If $i \in \mathrm{P}((T_i,\boldsymbol{B}_{-i}),\boldsymbol{b})$, we have $U_i((T_i,\boldsymbol{B}_{-i}),\boldsymbol{b}) = v_i - b_i$ and it is always greater than or equal to $U_i(\boldsymbol{B},\boldsymbol{b})$. Hence, in the rest of the proof, we consider the case that $T_i \subseteq B_i$ and $i \in \mathrm{Q}((T_i,\boldsymbol{B}_{-i}),\boldsymbol{b}) \cup \mathrm{R}((T_i,\boldsymbol{B}_{-i}),\boldsymbol{b})$. In this case, there exists an optimal solution $\boldsymbol{x}^* \in \Omega((T_i,\boldsymbol{B}_{-i}),\boldsymbol{b})$ satisfying $\boldsymbol{x}_i^* = 0$. Let $\overline{\boldsymbol{x}}$ be an optimal solution of BAP($\boldsymbol{B}$,$\boldsymbol{b}$). Since $T_i \subseteq B_i$, $\overline{\boldsymbol{x}}$ is also feasible for BAP($(T_i,\boldsymbol{B}_{-i})$,$\boldsymbol{b}$) and so $\boldsymbol{b}\boldsymbol{x}^* \geq \boldsymbol{b}\overline{\boldsymbol{x}}$. Conversely, since $x_i^* = 0$, $\boldsymbol{x}^*$ is also feasible for BAP($\boldsymbol{B}$,$\boldsymbol{b}$). Hence, we have $\boldsymbol{b}\overline{\boldsymbol{x}} \geq \boldsymbol{b}\boldsymbol{x}^*$ and so $\boldsymbol{b}\overline{\boldsymbol{x}} = \boldsymbol{b}\boldsymbol{x}^*$. From the above discussion, we find that the following three properties hold:

(i) $\Omega(\boldsymbol{B},\boldsymbol{b}) \subseteq \Omega((T_i,\boldsymbol{B}_{-i}),\boldsymbol{b})$,
(ii) $\{\boldsymbol{x} \in \Omega(\boldsymbol{B},\boldsymbol{b}) \mid x_i = 1\} \subseteq \{\boldsymbol{x} \in \Omega((T_i,\boldsymbol{B}_{-i}),\boldsymbol{b}) \mid x_i = 1\}$,
(iii) $\{\boldsymbol{x} \in \Omega(\boldsymbol{B},\boldsymbol{b}) \mid x_i = 0\} = \{\boldsymbol{x} \in \Omega((T_i,\boldsymbol{B}_{-i}),\boldsymbol{b}) \mid x_i = 0\}$.

Assume that $i \in \mathrm{R}((T_i, \boldsymbol{B}_{-i}), \boldsymbol{b})$. Then, we can assert that $i \in \mathrm{R}(\boldsymbol{B}, \boldsymbol{b})$. Otherwise, there exists $\overline{\boldsymbol{x}} \in \Omega(\boldsymbol{B}, \boldsymbol{b})$ such that $\overline{\boldsymbol{x}}_i = 1$. From (i), $\overline{\boldsymbol{x}}$ is also an optimal solution for $\mathrm{BAP}((T_i, \boldsymbol{B}_{-i}), \boldsymbol{b})$. However, it contradicts from $i \in \mathrm{R}((T_i, \boldsymbol{B}_{-i}), \boldsymbol{b})$. Hence, we have $i \in \mathrm{R}(\boldsymbol{B}, \boldsymbol{b})$, and so we have $U_i((T_i, \boldsymbol{B}_{-i}), \boldsymbol{b}) = U_i(\boldsymbol{B}, \boldsymbol{b}) = 0$.

Lastly, suppose that $i \in \mathrm{Q}((T_i, \boldsymbol{B}_{-i}), \boldsymbol{b})$. From (ii) and (iii), we have

$$
\begin{aligned}
U_i((T_i, \boldsymbol{B}_{-i}), \boldsymbol{b}) &= (v_i - b_i) \frac{|\{\boldsymbol{x} \in \Omega((T_i, \boldsymbol{B}_{-i}), \boldsymbol{b}) \mid x_i = 1\}|}{|\Omega((T_i, \boldsymbol{B}_{-i}), \boldsymbol{b})|} \\
&= (v_i - b_i) \frac{|\{\boldsymbol{x} \in \Omega((T_i, \boldsymbol{B}_{-i}), \boldsymbol{b}) \mid x_i = 1\}|}{|\{\boldsymbol{x} \in \Omega((T_i, \boldsymbol{B}_{-i}), \boldsymbol{b}) \mid x_i = 1\}| + |\{\boldsymbol{x} \in \Omega((T_i, \boldsymbol{B}_{-i}), \boldsymbol{b}) \mid x_i = 0\}|} \\
&\geq (v_i - b_i) \frac{|\{\boldsymbol{x} \in \Omega(\boldsymbol{B}, \boldsymbol{b}) \mid x_i = 1\}|}{|\{\boldsymbol{x} \in \Omega(\boldsymbol{B}, \boldsymbol{b}) \mid x_i = 1\}| + |\{\boldsymbol{x} \in \Omega(\boldsymbol{B}, \boldsymbol{b}) \mid x_i = 0\}|} \\
&= (v_i - b_i) \frac{|\{\boldsymbol{x} \in \Omega(\boldsymbol{B}, \boldsymbol{b}) \mid x_i = 1\}|}{|\Omega(\boldsymbol{B}, \boldsymbol{b})|} \\
&= U_i(\boldsymbol{B}, \boldsymbol{b}).
\end{aligned}
$$

Hence, we can conclude that $U_i((T_i, \boldsymbol{B}_{-i}), \boldsymbol{b}) \geq U_i(\boldsymbol{B}, \boldsymbol{b})$ for any B_i. ||

Proof of Lemma 3.

When $\mathrm{P}(\boldsymbol{B}, \boldsymbol{v}) = \emptyset$, both $\mathcal{F}_\varepsilon(\boldsymbol{B}, \boldsymbol{v})$ and $\overline{\mathcal{F}}(\boldsymbol{B}, \boldsymbol{v})$ are singleton, and so it is obvious. In the rest of this proof, we assume that $\mathrm{P}(\boldsymbol{B}, \boldsymbol{v}) \neq \emptyset$.

In this proof, we define the affine subspace H by

$$
H = \{\boldsymbol{b} \in \mathrm{R}^N \mid b_i = v_i \ (\forall i \in \mathrm{R}(\boldsymbol{B}, \boldsymbol{v}) \cup \mathrm{Q}(\boldsymbol{B}, \boldsymbol{v}))\}.
$$

First, we show that the minimum affine subspace including $\overline{\mathcal{F}}(\boldsymbol{B}, \boldsymbol{v})$ is H. Clearly, $\overline{\mathcal{F}}(\boldsymbol{B}, \boldsymbol{v}) \subseteq H$. For any passed bidder $i \in \mathrm{P}(\boldsymbol{B}, \boldsymbol{v})$, if η is a sufficiently small positive number, then $\boldsymbol{v} - \eta \boldsymbol{u}_i \in \overline{\mathcal{F}}(\boldsymbol{B}, \boldsymbol{v})$ where $\boldsymbol{u}_i$ is the ith unit vector. Since the minimum affine subspace including $\{\boldsymbol{v}\} \cup \{\boldsymbol{v} - \eta \boldsymbol{u}_i \mid i \in \mathrm{P}(\boldsymbol{B}, \boldsymbol{v})\}$ is H, the minimum affine subspace including $\overline{\mathcal{F}}(\boldsymbol{B}, \boldsymbol{v})$ is also H.

We define the set $\mathcal{F}'$ by

$$
\mathcal{F}' \stackrel{\mathrm{def.}}{=} \left\{ \boldsymbol{b} \in \mathrm{R}^N \;\middle|\; \begin{array}{l} b_i = v_i \ \ (\forall i \in \mathrm{R}(\boldsymbol{B}, \boldsymbol{v}) \cup \mathrm{Q}(\boldsymbol{B}, \boldsymbol{v})), \\ b_i < v_i \ \ (\forall i \in \mathrm{P}(\boldsymbol{B}, \boldsymbol{v})), \\ \Omega(\boldsymbol{B}, \boldsymbol{b}) = \Omega(\boldsymbol{B}, \boldsymbol{v}) \end{array} \right\}.
$$

Then, we only need to show that $\mathcal{F}' = \mathrm{int}\overline{\mathcal{F}}(\boldsymbol{B}, \boldsymbol{v})$. For any point $\boldsymbol{b}_1 \in \mathcal{F}'$, we consider the open ball $\mathrm{Ball}(\boldsymbol{b}_1, \eta_1)$ with center $\boldsymbol{b}_1$ and radius η_1. Assume that η_1 is a sufficiently small positive number. Then, for any vector $\boldsymbol{b}' \in \mathrm{Ball}(\boldsymbol{b}_1, \eta_1) \cap H$, $\boldsymbol{b}' \in \mathcal{F}' \subseteq \overline{\mathcal{F}}(\boldsymbol{B}, \boldsymbol{v})$ and so $\mathrm{Ball}(\boldsymbol{b}_1, \eta_1) \cap H \subseteq \overline{\mathcal{F}}(\boldsymbol{B}, \boldsymbol{v})$. It implies that $\mathcal{F}' \subseteq \mathrm{int}\overline{\mathcal{F}}(\boldsymbol{B}, \boldsymbol{v})$. Let $\boldsymbol{b}_2$ be a point in $\mathrm{int}\overline{\mathcal{F}}(\boldsymbol{B}, \boldsymbol{v})$. Then there exists an open ball $\mathrm{Ball}(\boldsymbol{b}_2, \eta_2)$ satisfying that $\mathrm{Ball}(\boldsymbol{b}_2, \eta_2) \cap H \subseteq \overline{\mathcal{F}}(\boldsymbol{B}, \boldsymbol{v})$. Then, it is clear that for any passed bidder $i \in \mathrm{P}(\boldsymbol{B}, \boldsymbol{v})$, ith element of $\boldsymbol{b}_2$ is strictly less than v_i. Lastly, we show that $\Omega(\boldsymbol{B}, \boldsymbol{b}_2) = \Omega(\boldsymbol{B}, \boldsymbol{v})$. Assume on the contrary that $\exists \boldsymbol{x}^* \in \Omega(\boldsymbol{B}, \boldsymbol{b}_2) \setminus \Omega(\boldsymbol{B}, \boldsymbol{v})$. If $x^*_i = 1$ for all $i \in \mathrm{P}(\boldsymbol{B}, \boldsymbol{v})$, then $\boldsymbol{x}^* \in \Omega(\boldsymbol{B}, \boldsymbol{v})$

and it is a contradiction. Thus, $\exists i' \in \mathrm{P}(\boldsymbol{B}, \boldsymbol{v})$, $x^*_{i'} = 0$. Let us consider the vector $\boldsymbol{b}_2 - (1/2)\eta_2 \boldsymbol{u}_{i'}$. Since $\boldsymbol{b}_2 - (1/2)\eta_2 \boldsymbol{u}_{i'} \in \mathrm{Ball}(\boldsymbol{b}_2, \eta_2) \cap H$, it is clear that $\Omega(\boldsymbol{B}, \boldsymbol{b}_2 - (1/2)\eta_2 \boldsymbol{u}_{i'}) \supseteq \Omega(\boldsymbol{B}.\boldsymbol{v})$. Denote a solution in $\Omega(\boldsymbol{B}, \boldsymbol{v})$ by $\boldsymbol{x}'$. Then, $\boldsymbol{x}^* \in \Omega(\boldsymbol{B}, \boldsymbol{b}_2) \supseteq \Omega(\boldsymbol{B}, \boldsymbol{v}) \ni \boldsymbol{x}'$ implies the inequality

$$(\boldsymbol{b}_2 - (1/2)\eta_2 \boldsymbol{u}_{i'}) \cdot \boldsymbol{x}' < \boldsymbol{b}_2 \cdot \boldsymbol{x}' = \boldsymbol{b}_2 \cdot \boldsymbol{x}^* = (\boldsymbol{b}_2 - (1/2)\eta_2 \boldsymbol{u}_{i'}) \cdot \boldsymbol{x}^*$$

and so $\Omega(\boldsymbol{b}_2 - (1/2)\eta_2 \boldsymbol{u}_{i'}) \not\ni \boldsymbol{x}' \in \Omega(\boldsymbol{B}, \boldsymbol{v})$. Contradiction. ||

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